0.1. **Linear Maps and Linear Transformation.** Geometrically speaking, linear transformation is the transformation that keeping the parallel lines and origin. Rotation around origin, reflection with respect to a subspace passes through origin are all linear transformations.

**Rotation by 90 degree:** Parallel lines keeps parallel after rotation.

The blue lines are obtained by rotating the black parallel lines. We observe that the relation of parallel will kept by this kind of action. This is a linear transformation.

Imagin that you have a cube dice put on the desk, you now would like to take photo of it. And the following is its picture. If you are far enough and the cube is small enough, then the parallel lines on the cube will goes to parallel lines. This is a linear map, since one linear space is our living space, the target space is this the plane of this page.

**Definition 0.1.1**

Suppose $V_1, V_2$ are two right linear space over $F$. A map $T : V_1 \to V_2$ is called a **linear map** if for any vector $v, w \in V_1$, and any scalar $\lambda$, we have **linearity**:

1. $T(v + w) = Tv + Tw$ (**Distributivity**)
2. $T(\lambda v) = \lambda (Tv)$ (**Associativity**)

We call the space $V_1$ **Domain Space**, and space $V_2$ the **Target Space**. In the case that Domain space and Target Space are the same, we often call it **Linear Transformation**.

Algebraically speaking, linearity is distributivity together with associativity.

If $V$ is a left linear space, we write $T$ on the right, and those conditions became as $(v + w)T = vT + wT$ and $(\lambda v)T = \lambda(vT)$.

0.2. **Calculation of Linear Transformation.**
In vector spaces, the linear map are those map who keeping the parallel lines. Rotation, Reflection, Projection. **How to calculate the coordinate of vectors after linear transformation?**

The following picture demonstrate a linear transformation of rotating $\theta$ degree.

We call this linear transformation as $T$. The graph rotated by angle $\arctan\left(\frac{3}{4}\right)$. Since linear transformation keeps the linear combination. It will be suffice to us to understand it soely on basis. When writing down the coordinates of $Te_1, Te_2$ respectively, we have

$$T \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} Te_1 \\ Te_2 \end{pmatrix}$$

$$= \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$$
Let us given \( \mathbf{v} = (e_1 \ e_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} \). How to calculate \( T\mathbf{v} \)?

Because \( T \) is linear, so we have

\[
T\mathbf{v} = T(e_1 \ e_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
= (e_1 \ e_2) \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
= (e_1 \ e_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
= e_1 + 2e_2
\]

**Summary**

In the above example. We applied linear transformation on basis and end up with

\[
T(e_1 \ e_2) = (e_1 \ e_2) \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}
\]

To know value of any \( T\mathbf{v} \) is just by doing substitution and end up with multiplying the matrix on coordinates.
Definition 0.2.2

Suppose $T : V \rightarrow W$ is a linear map, and $(e_1 \ e_2 \ \cdots \ e_n)$ is basis for $V$, $(\epsilon_1 \ \epsilon_2 \ \cdots \ \epsilon_m)$ is basis for $W$. Then there exists a unique matrix $P$, with size $n \times m$, such that

$$T(e_1 \ e_2 \ \cdots \ e_n) = (\epsilon_1 \ \epsilon_2 \ \cdots \ \epsilon_m)P$$

Then this matrix $P$ is called the **matrix representation of $T$ with respect to basis $(e_1 \ e_2 \ \cdots \ e_n)$ of Domain Space and basis $(\epsilon_1 \ \epsilon_2 \ \cdots \ \epsilon_m)$ of Target Space**

0.2.1. Change of basis.

Suppose we have linear space $V$ with basis $(e_1 \ e_2 \ e_3)$ and $W$ with basis $(\epsilon_1 \ \epsilon_2)$ respectively. $T : V \rightarrow W$ is a linear map and suppose we know following.

$$(v_1 \ v_2 \ v_3) = (e_1 \ e_2 \ e_3) \begin{pmatrix} 2 & 1 & 4 \\ 5 & 2 & 7 \\ 6 & 1 & 1 \end{pmatrix}; \quad (w_1 \ w_2) = (\epsilon_1 \ \epsilon_2) \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix};$$

\[
\begin{align*}
Te_1 &= 3\epsilon_1 + 3\epsilon_2 \\
Te_2 &= 3\epsilon_1 + 2\epsilon_2 \\
Te_3 &= \epsilon_1 + 3\epsilon_2
\end{align*}
\]

Now we want to represent $T$ with respect to bases $(v_1 \ v_2 \ v_3)$ and $(w_1 \ w_2)$. That is easy. We know

$$T(e_1 \ e_2 \ e_3) = (\epsilon_1 \ \epsilon_2) \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 3 \end{pmatrix}$$

and

$$(\epsilon_1 \ \epsilon_2) = (w_1 \ w_2) \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}^{-1}$$

We do the following steps

$$T(v_1 \ v_2 \ v_3) = T(e_1 \ e_2 \ e_3) \begin{pmatrix} 2 & 1 & 4 \\ 5 & 2 & 7 \\ 6 & 1 & 1 \end{pmatrix}$$

$$= (\epsilon_1 \ \epsilon_2) \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 \\ 5 & 2 & 7 \\ 6 & 1 & 1 \end{pmatrix}$$

$$= (w_1 \ w_2) \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 \\ 5 & 2 & 7 \\ 6 & 1 & 1 \end{pmatrix}$$

$$= (w_1 \ w_2) \begin{pmatrix} -1 & 10 & 54 \\ 8 & -10 & -59 \end{pmatrix}$$

So that means the matrix representation of $T$ with respect to $(v_1 \ v_2 \ v_3)$ and $(w_1 \ w_2)$ is
In general, there is a change of basis formula.

**Proposition 0.1**

Let \( T : V \rightarrow W \) be a linear map. Suppose \((v_1, \ldots, v_n)\) and \((e_1, \ldots, e_n)\) are bases of \(V\) and \((w_1, \ldots, w_n)\) and \((\epsilon_1, \ldots, \epsilon_m)\) are bases of \(W\). Suppose

\[
\begin{pmatrix}
  v_1 & \cdots & v_n \\
\end{pmatrix}
= \begin{pmatrix}
  e_1 & \cdots & e_n \\
\end{pmatrix} P
\]

\[
\begin{pmatrix}
  w_1 & \cdots & w_m \\
\end{pmatrix}
= \begin{pmatrix}
  \epsilon_1 & \cdots & \epsilon_m \\
\end{pmatrix} Q
\]

Then

\[
T\left(\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right) = \left(\begin{array}{c} w_1 \\ \vdots \\ w_m \end{array}\right) Q^{-1} AP
\]

**Proof.**

\[
T\left(\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right) = T\left(\begin{array}{c} e_1 \\ \vdots \\ e_n \end{array}\right) P
\]

\[
= \left(\begin{array}{c} e_1 \\ \vdots \\ e_m \end{array}\right) AP
\]

\[
= \left(\begin{array}{c} w_1 \\ \vdots \\ w_m \end{array}\right) Q^{-1} AP
\]

\[\Box\]

In particular, in the case of \(V = W\). We have the following statement

**Corollary 0.1**

Let \( T : V \rightarrow V \) be a linear transformation. Suppose \((v_1, \ldots, v_n)\) and \((e_1, \ldots, e_n)\) are bases of \(V\) and

\[
\begin{pmatrix}
  v_1 & \cdots & v_n \\
\end{pmatrix}
= \begin{pmatrix}
  e_1 & \cdots & e_n \\
\end{pmatrix} P
\]

Then

\[
T\left(\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right) = \left(\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right) P^{-1} AP
\]

In linear spaces of column matrices, which is a right linear space. Any linear transformation is the same as left multiplying a matrix. So matrices could viewed as linear map between right linear spaces of column matrices. We demonstrate this by showing following example.
Suppose \( V = M_{2 \times 1}(\mathbb{R}) \), \( W = M_{3 \times 1}(\mathbb{R}) \), And suppose we have a linear transformation \( T : V \rightarrow W \) defined by

\[
T\left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
T\left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

Find the matrix representation of \( T \) with respect to the basis \( \left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \) in domain space and basis \( \left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \) (0 0 1) in target space

Answer: We proceed like following: we test our linear transformation on each of the vectors of basis:

\[
T\left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times 1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times 1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times 0
\]

\[
T\left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times 0 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times 1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times 1
\]

Then we put our work into matrix, we get

\[
T\left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}
\]

So the matrix representation of \( T \) with respect to that basis is \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \).

Now let’s look what is \( T\left( \begin{pmatrix} a \\ b \end{pmatrix} \right) \)? Because in the natural basis, the coordinate of column matrix is just this matrix itself. In this way, we have
\[
T\left(\begin{array}{c}
a \\
b \\
\end{array}\right)
= T\left(\begin{array}{c}
1 \\
0 \\
0 \\
1 \\
\end{array}\right)\left(\begin{array}{c}
a \\
b \\
\end{array}\right)
= \left(\begin{array}{c}
1 \\
0 \\
0 \\
0 \\
1 \\
\end{array}\right)\left(\begin{array}{c}
1 \\
0 \\
1 \\
1 \\
0 \\
1 \\
\end{array}\right)\left(\begin{array}{c}
a \\
b \\
\end{array}\right)
= \left(\begin{array}{c}
1 \\
0 \\
1 \\
0 \\
\end{array}\right)\left(\begin{array}{c}
a \\
b \\
\end{array}\right)
\]

So we see in this example, the linear map that took place from column matrices to column matrices is just realized by matrix multiplication. And this matrix is obtained by represent the linear map with respect to those basis.

![Warning]

In linear space of row matrices, which is a left linear space. Any linear transformation is the same as right multiplying a matrix. So matrices could viewed as linear map between left linear spaces of row matrices. Example dismissed.

Suppose \( V = P^2_\mathbb{R} = \{ x^2 a + xb + c | a, b, c \in \mathbb{R} \} \) is the linear space of polynomial with real coefficient with degree at most 2. \( W = P^2_t = \{ t^2 a + tb + c | a, b, c \in \mathbb{R} \} \) \( T : V \rightarrow W \) is a linear map defined by evaluating by plug in \( x = t + 2 \)

(1) with the basis \( \{ x^2 \ x \ 1 \} \) in \( V \), and (1) in \( \mathbb{R} \), write down the matrix that representing the linear map.

Answer: We test it on each of the basis:

\[
T(x^2) = (t + 2)^2 = \left(\begin{array}{c}
t^2 \\
t \\
1 \\
\end{array}\right)\left(\begin{array}{c}
1 \\
4 \\
4 \\
\end{array}\right)
\]

\[
T(x) = t + 2 = \left(\begin{array}{c}
t^2 \\
t \\
1 \\
\end{array}\right)\left(\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}\right)
\]

\[
T(1) = 1 = \left(\begin{array}{c}
t^2 \\
t \\
1 \\
\end{array}\right)\left(\begin{array}{c}
0 \\
0 \\
1 \\
\end{array}\right)
\]

So we have
\[ T( x^2 \ x \ 1 ) = ( t^2 \ t \ 1 ) \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \]

(2) Suppose \( f = (x + 1)(x + 3) \in V \) has coordinate \( \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \) with respect to basis \( ( x^2 \ x \ 1 ) \). what is the coordinate of \( Tv \) with respect to basis \( ( t^2 \ t \ 1 ) \)?

First Method (Directly by definition)
By definition
\[
T((x + 1)(x + 3)) = (t + 3)(t + 5) = t^2 + 8 + 15 = ( t^2 \ t \ 1 ) \begin{pmatrix} 1 \\ 8 \\ 15 \end{pmatrix}
\]

Second Method (Standard linear algebra method)
By using the matrix representation
\[
T((x + 1)(x + 3)) = T(x^2 + 4x + 3)
\]
\[
= T( x^2 \ x \ 1 ) \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}
\]
\[
= ( t^2 \ t \ 1 ) \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}
\]
\[
= ( t^2 \ t \ 1 ) \begin{pmatrix} 1 \\ 8 \\ 15 \end{pmatrix}
\]

Linear algebra have tremendous application in combinatorial problems. Simply most of them is linear.
Let’s start with the following problem.

There are 3 brothers. Namely, A,B,C. They have some money originally. Suppose there is neither income and costs. And they like to share their money. At the end of every month, each of them will split his money into two equal parts and give it to other two. For example. Suppose A has 300 dollars at first month. At second month, A with give 150 to B and give 150 to C.

(1) Suppose in a month, A has $1000, B has $ 500, C has $ 200, what is the money of them next month.

Answer: Of course we can do it directly, As A will receive half of B and half of C, so A would have 350. B will receive half of A and half of C, so B would have 600, C would receive half of A and half of B, so C would have 750.

But, we want to use higher understanding to this process. If we view all possible status of the money each one has as a linear space. OK, then it is 3 dimensional. And at the end of every month, the status changes, and this change, is linear. So it is the same as saying
every month we do an linear transformation of current status, and the output is the status next month.

Let V be the linear space of the current account status of A, B, C. We choose a basis for V

\[ e_1 = \text{a dollar that belongs to A} \]
\[ e_2 = \text{a dollar that belongs to B} \]
\[ e_3 = \text{a dollar that belongs to C} \]

And let T be the operator that changes the status to the next month. What is \( T e_1 \), \( T e_2 \), \( T e_3 \) respectively?

Because A will split every dollar of him and give it to B and C. So a dollar belongs to A now become two quarter belongs to B and two quarter belongs to C. That is

\[ T e_1 = \frac{1}{2} e_2 + \frac{1}{2} e_3 \]

And we write everything down, that is

\[
\begin{align*}
T e_1 &= \frac{1}{2} e_2 + \frac{1}{2} e_3 \\
T e_2 &= \frac{1}{2} e_1 + \frac{1}{2} e_3 \\
T e_3 &= \frac{1}{2} e_1 + \frac{1}{2} e_2
\end{align*}
\]

Now put our work into matrix, that is

\[
T \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}
\]

Because A has 1000, B has 500, C has 200. This status corresponding to vectors

\[
\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 1000 \\ 500 \\ 200 \end{pmatrix}
\]

And plug in the matrix representation form of T, the linear transformation act on this would result matrix multiplication on the coordinates:

\[
T \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 1000 \\ 500 \\ 200 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1000 \\ 500 \\ 200 \end{pmatrix}
\]

\[
= \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 350 \\ 600 \\ 750 \end{pmatrix}
\]

\[
= 350 e_1 + 600 e_2 + 750 e_3
\]

That means the status of next month. 350 dollars belongs to A, 600 dollars belongs to B, and 750 dollars belongs to C.

In this example, the matrix \( \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix} \) is the matrix representation of T with respect to the basis.
Do something to show composition of two linear transformation is multiplying those matrix together.